J-Noetherian Bezout domain which are not of stable range 1. A Bezout ring of stable range 2 which has square stable range 1

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Plan

A J-Noetherian Bezout domain which are not of stable range 1.

- *Keywords*: J-Noetherian ring; Bezout ring; adequate ring; elementary divisor ring; stable range; almost stable range; neat range.

B A Bezout ring of stable range 2 which has square stable range 1.

- *Keywords*: Hermitian ring; elementary divisor ring; stable range 1; stable range 2; square stable range 1; Toeplitz matrix; duo ring; quasi-duo ring.

J-Noetherian Bezout domain which are not of stable range 1

All rings considered will be commutative and have identity.

The notion of a stable range was useful in modern research on theory of diagonalization of matrices.

Definition

A ring R is a **ring of stable range 1** if for any $a, b \in R$ such that $aR + bR = R$ we have $(a + bt)R = R$ for some $t \in R$.

Studying an elementary divisor ring W. McGovern has introduced the concept of **ring of almost stable range 1** as a ring whose proper homomorphic images all have stable range 1 [McGovern,2007].

By [McGovern,2007] a ring of stable range 1 is a ring of almost stable range 1. At the same time not every element of stable range 1 is an element of almost stable range 1.

W. McGovern, Bezout rings with almost stable range 1 are elementary divisor rings, *J. Pure Appl. Algebra* **212** (2007) 340–348.

Definition

An element a is an **element of stable range 1** if for any $b \in R$ such that $aR + bR = R$ we have $a + bt$ is an invertible element for some $t \in R$.

Definition

An element $a \in R$ is an **element of almost stable range 1** if R/aR is a ring of stable range 1.

If $R = \mathbb{Z} \times \mathbb{Z}$, then $e = (1, 0)$ is element of stable range 1, but $R/eR \cong \mathbb{Z}$ is not a ring of stable range 1.

The problem of finding the element of almost stable range 1 in rings which are not rings of stable range 1 is required, in accordance with the above and considerations in [Zabavsky, 2017].

On the basis of theory comaximal factorization we prove that in any J-Noetherian Bezout ring which are not of stable range 1 there exist a nonunit element of almost stable range 1.

B. V. Zabavsky, Conditions for stable range of an elementary divisor rings, *Comm. Alg.* **45**(9) (2017) 4062–4066.

By a **Bezout ring** we mean a ring in which all finitely generated ideals are principal.

By a J-**ideal** of R we mean an intersection of maximal ideals of R.

A ring R is J-**Noetherian** provided R has maximum condition of J-ideals.

Unique factorization domains are, of course, integral domains in which every nonzero nonunit element has a unique factorization (up to order and associates) into irreducible elements, or atoms. Now UFDs can also be characterized by the property that every nonzero nonunit is a product of principal primes or equivalently that every nonzero nonunit has the form $p_1^{\alpha_1} \cdot \ldots \cdot p_n^{\alpha_n}$, where p_1, ..., p_n are non-associate principal primes and each $\alpha_i \geq 1$. Each of the $p_i^{\alpha_i}$, in addition to being a power of a prime, has other properties, each of which is subject to generalization. For example, each $p_i^{\alpha_i}$ is primary and the $p_i^{\alpha_i}$ are pairwise comaximal. There exist various generalizations of a (unique) factorization into prime powers in integral domains [Brewer, Heinzer, 2002].

J. W. Brewer and W. J. Heinzer, On decomposing ideals into products of comaximal ideals, *Comm. Algebra* **30**(12) (2002) 5999–6010.

We consider the comaximal factorization introduced by McAdam and Swan [McAdam, Swan, 2004].

They defined a nonzero nonunit element d of an integral domain R to be **pseudo-irreducible** (**pseudo-prime**) if $d = ab$ ($abR \supset dR$) for comaximal a and b implies that a or b is a unit ($aR \supset dR$ or $bR \supset dR$).

A factorization $d = d_i \dots d_n$ is a **complete comaximal factorization** if each d_i is a nonzero nonunit pseudo-irreducible and the d_i's are pairwise comaximal. The integral domain is a **comaximal factorization domain** (CFD) if each nonzero nonunit has a complete comaximal factorization.

S. McAdam and R. Swan, Unique comaximal factorization, *J. Algebra* **276** (2004) 180–192.

Let us start with the following Henriksen example [Henriksen, 1955]

$$R = \{z_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \mid z_0 \in \mathbb{Z},\ a_i \in \mathbb{Q},\ i = 1, 2, \ldots\}.$$

This domain is a two-dimensional Bezout domain having a unique prime ideal $J(R)$ (Jacobson radical) of height one and having infinitely many maximal ideals corresponding to the maximal ideals of $\mathbb{Z}$.

The elements of $J(R)$ are contained in infinitely many maximal ideals of R while the elements of $R \setminus J(R)$ are contained in only finitely many prime ideals.

M. Henriksen, Some remarks about elementary divisor rings, *Michigan Math. J.* **3** (1955/56) 159–163.

Henriksen's Example is an example of a ring in which every nonzero nonunit element has only finitely many prime ideals minimal over it. By [McAdam, Swan, 2004], Lemma 1.1.(i), in Henriksen's example every nonzero nonunit element has a complete comaximal factorization. The element of $J(R)$ has a complete comaximal factorization cx^n, where $c \in \mathbb{Q}$, $n \geq 1$ (cx^n is a pseudo-irreducible element). By [McAdam, Swan, 2004], Lemma 3.2, any nonzero element of $J(R)$ is pseudo-irreducible. The elements of $R \setminus J(R)$ are contained in only finitely many prime ideals and have a complete comaximal factorization corresponding their factorization in $\mathbb{Z}$. We will notice $J(R)$ is not generated by x, since $\frac{1}{2}x \in J(R)$ and $\frac{1}{2}x \notin xR$.

S. McAdam and R. Swan, Unique comaximal factorization, *J. Algebra* **276** (2004) 180–192.

It is shown in [McAdam, Swan, 2004] that an integral domain R is a complete comaximal factorization if either 1) each nonzero nonunit of R has only finitely many minimal primes or 2) each nonzero nonunit of R is contained in only finitely many maximal ideals.

S. McAdam and R. Swan, Unique comaximal factorization, *J. Algebra* **276** (2004) 180–192.

For a commutative J-Noetherian Bezout domain R this condition is equivalent to a condition that every nonzero nonunit element of R has only finitely many prime ideals minimal over it [Estes, Ohm, 1967].

Lemma 2.1

Let R be a Bezout domain and a be a nonzero nonunit element of R. Then a is pseudo-irreducible if and only if $\overline{R} = R/aR$ is connected (i.e. its only idempotents are zero and 1).

D. Estes and J. Ohm, Stable range in commutative rings, *J. Algebra* **7** (1967) 343–362.

We will notice that in a local ring, every nonzero nonunit element is pseudo-irreducible.

If a is a element of domain R where $|\,minspec\, a\,| = 1$, then a is also pseudo-irreducible (denote by $minspec\, a$ the set of prime ideals minimal over a).

In Henriksen's example

$$R = \{z_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \mid z_0 \in \mathbb{Z},\ a_i \in \mathbb{Q},\ i = 1, 2, \dots\},$$

x is pseudo-irreducible.

Definition

A commutative ring R is called an **elementary divisor ring** [Kaplansky,1949] if for an arbitrary matrix A of order $n \times m$ over R there exist invertible matrices $P \in GL_n(R)$ and $Q \in GL_m(R)$ such that

(1) $PAQ = D$ is diagonal matrix, $D = (d_{ii})$;

(2) $d_{i+1,i+1}R \subset d_{ii}R$.

I. Kaplansky, Elementary divisors and modules, *Trans. Amer. Math. Soc.* **66** (1949) 464–491.

By [Shores,Wiegand,1974], we have the following result.

Theorem 2.1

Every J-Noetherian Bezout ring is an elementary divisor ring.

T. Shores and R. Wiegand, Some criteria for Hermite rings and elementary divisor rings, *Can J. Math.* **XXVI**(6) (1974) 1380–1383.

By [Zabavsky, Bokhonko, 2017], for a Bezout domain we have the following result.

Theorem 2.2

Let R be a Bezout domain. The following two condition are equivalent:

(1) R is an elementary divisor ring;

(2) for any elements $x, y, z, t \in R$ such $xR + yR = R$ and $zR + tR = R$ there exists an element $\lambda \in R$ such that $x + \lambda y = r \cdot s$, where $rR + zR = R$, $sR + tR = R$ and $rR + sR = R$.

B. V. Zabavsky and V. V. Bokhonko, A criterion of elementary divisor domain for distributive domains, *Algebra and Discrete Math.* **23**(1) (2017) 1–6.

Definition

Let R be a Bezout domain. An element $a \in R$ is called a **neat element** if for every elements $b, c \in R$ such that $bR + cR = R$ there exist $r, s \in R$ such that $a = rs$ where $rR + bR = R$, $sR + cR = R$ and $rR + sR = R$.

Definition

A Bezout domain is said to be of **neat range 1** if for any $c, b \in R$ such that $cR + bR = R$ there exists $t \in R$ such that $a + bt$ is a neat element.

According to Theorem 2.2 we will obtain the following result.

Theorem 2.3

A commutative Bezout domain R is an elementary divisor domain if and only if R is a ring of neat range 1.

Theorem 2.4

A nonunit divisor of a neat element of a commutative Bezout domain is a neat element.

Theorem 2.5

Let R be a J-Noetherian Bezout domain which is not a ring of stable range 1. Then in R there exists an element $a \in R$ such that R/aR is a local ring.

By [Zabavsky, 2014], any adequate element of a commutative Bezout ring is a neat element.

Definition

An element a of a domain R is said to be **adequate** , if for every element $b \in R$ there exist elements $r, s \in R$ such that:

(1) $a = rs$;

(2) $rR + bR = R$;

(3) $\hat{s}R + bR \neq R$ for any $\hat{s} \in R$ such that $sR \subset \hat{s}R \neq R$.

B. V. Zabavsky, Diagonal reduction of matrices over finite stable range rings, *Mat. Stud.* **41** (2014) 101–108.

Definition

A domain R is called **adequate** if every nonzero element of R is adequate [Larsen, Levis, Shores, 1974].

The most trivial examples of adequate elements are units, atoms in a ring, and also square-free elements.

M. Larsen, W. Levis and T. Shores, Elementary divisor rings and finitely presented modules, *Trans. Amer. Math. Soc.* **187** (1974) 231–248.

Henriksen observed that in an adequate domain every nonzero prime ideal is contained in an unique maximal ideal [Henriksen, 1955].

Theorem 2.6

Let R be a commutative Bezout element and a is non-zero nonunit element of R. If R/aR is local ring, then a is an adequate element.

M. Henriksen, Some remarks about elementary divisor rings, *Michigan Math. J.* **3** (1955/56) 159–163.

Theorem 2.7

Let R be a J-Noetherian Bezout domain which is not a ring of stable range 1. Then in R there exists a nonunit adequate element.

Theorem 2.8

Let R be a Bezout domain in which every nonzero nonunit element has only finitely many prime ideals minimal over it. Then the factor ring R/aR is the finite direct sum of valuation rings.

A minor modification of the proof of Theorem 2.8 gives us the following result.

Theorem 2.9

Let R be a commutative Bezout domain in which any nonzero prime ideal is contained in a finite set of maximal ideals. Then for any nonzero element $a \in R$ such that the set $minspec(aR)$ is finite, the factor ring $\overline{R} = R/aR$ is a finite direct sum of semilocal rings.

Definition

A ring R is said to be **everywhere adequate** if any element of R is adequate.

Note that, as shown above, in the case of a commutative ring, which is a finite direct sum of valuation rings, any element of the ring (in particular zero) is adequate, i.e. this ring is everywhere adequate.

We obtain the result from the previous theorems.

Theorem 2.10

Let R be a Bezout domain in which every nonzero nonunit element has only finitely many prime ideals minimal over it. Then the factor ring R/aR is everywhere adequate if and only if R is a finite direct sum of valuation rings.

Recall that a commutative ring in which every nonzero element is an element of almost stable range 1 is called a **ring of almost stable range 1**.

The first example of a ring of almost stable range 1 is a ring of stable range 1. Also, every commutative principal ideal ring which is not a ring of stable range 1 (for example, the ring of integers) is a ring of almost stable range 1 which is not a ring of stable range 1.

We note that a semilocal ring is an example of a ring of stable range 1. Moreover, a direct sum of rings of stable range 1 is a ring of stable range 1.

Theorem 2.11

Let R be a Bezout domain in which every nonzero nonunit element has only finitely many prime ideals minimal over it and any nonzero prime ideal $spec(aR)$ is contained in a finite set of maximal ideals. Then a is an element of almost stable range 1.

Open Question

Is it true that every commutative Bezout domain in which any non-zero prime ideal is contained in a finite set of maximal ideals is an elementary divisor ring?

A Bezout ring of stable range 2 which has square stable range 1

The notion of a stable range of a ring was introduced by H. Bass, and became especially popular because of its various applications to the problem of cancellation and substitution of modules.
Let us say that a module A satisfies the power-cancellation property if for all modules B and C, $A \oplus B \cong A \oplus C$ implies that $B^n \cong C^n$ for some positive integer n (here B^n denotes the direct sum of n copies of B).

Let us say that a right R-module A has the **power-substitution property** if given any right R-module decomposition $M = A_1 \oplus B_1 = A_2 \oplus B_2$ which each $A_i \cong A$, there exist a positive integer n and a submodule $C \subseteq M^n$ such that $M^n = C \oplus B_1^n = C \oplus B_2^n$.

K. Goodearl [Goodearl, 1976] pointed out that a commutative ring R has the power-substitution property if and only if R is of (right) power stable range 1, i.e. if $aR + bR = R$ than $(a^n + bx)R = R$ for some $x \in R$ and some integer $n \geq 2$ depending on $a, b \in R$.

K. R. Goodearl, Power-cancellation of groups and modules, *Pacific J. Math.* **64** (1976) 487–411.

Recall that a ring R is said to have 1 in **the stable range** provided that whenever $ax + b = 1$ in R, there exists $y \in R$ such that $a + by$ is a unit in R.

The following Warfield's theorem shows that 1 in the stable range is equivalent to a substitution property.

Theorem 3.1

Let A be a right R-module, and set $E = \mathrm{End}_R(A)$. Then E has 1 in the stable range if and only if for any right R-module decomposition $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with each $A_i \cong A$, there exists a submodule $C \subseteq M$ such that $M = C \oplus B_1 = C \oplus B_2$.

R. B. Warfield, Jr., *Notes on cancellation, stable range, and related topics*, Univ. of Washington, (August, 1975).

K. Goodearl pointed out to us the following result.

Proposition 3.1

Let R be a commutative ring which has 2 in the stable range. If R satisfies right power-substitution, then so does $M_n(R)$, for all n. [Goodearl, 1976]

K. R. Goodearl, Power-cancellation of groups and modules, *Pacific J. Math.* **64** (1976) 487–411.

A ring R is said to have 2 in the stable range if for any $a_1, \ldots, a_r \in R$ where $r \geq 3$ such that $a_1 R + \cdots + a_r R = R$, there exist elements $b_1, \ldots, b_{r-1} \in R$ such that

$$(a_1 + a_r b_1)R + (a_2 + a_r b_2)R + \cdots + (a_{r-1} + a_r b_{r-1})R = R.$$

Our goal is to study certain algebraic versions of the notion of stable range 1. We study a Bezout ring which has 2 in the stable range and which is a ring square stable range 1.

A ring R is said to have (right) **square stable range 1** (written $ssr(R) = 1$) if $aR + bR = R$ for any $a, b \in R$ implies that $a^2 + bx$ is an invertible element of R for some $x \in R$.

Considering the problem of factorizing the matrix $\left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right)$ into a product of two Toeplitz matrices. D. Khurana, T.Y. Lam and Zhou Wang were led to ask go units of the form $a^2 + bx$ given that $aR + bR = R$.

Obviously, a commutative ring which has 1 in the stable range is a ring which has (right) square stable range 1, but not vice versa in general. Examples of rings which have (right) square stable range 1 are rings of continuous real-valued functions on topological spaces and real holomorphy rings in formally real fields [Khurana, Lam, Wang, 2011].

D. Khurana, T. Y. Lam and Zh. Wang, Rings of square stable range one, *J. Algebra* **338** (2011) 122–143.

Proposition 3.2

For any ring R with $ssr(R) = 1$, we have that R is right quasi-duo (i.e. R is a ring in which every maximal right ideal is an ideal). [Khurana, Lam, Wang, 2011]

D. Khurana, T. Y. Lam and Zh. Wang, Rings of square stable range one, *J. Algebra* **338** (2011) 122–143.

We say that matrices A and B over a ring R are **equivalent** if there exist invertible matrices P and Q of appropriate sizes such that $B = PAQ$. If for a matrix A there exists a diagonal matrix $D = diag(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r, 0, \ldots, 0)$ such that A and D are equivalent and $R\varepsilon_{i+1}R \subseteq \varepsilon_i R \cap R\varepsilon_i$ for every i then we say that the matrix A has a **canonical diagonal reduction**. A ring R is called an **elementary divisor ring** if every matrix over R has a canonical diagonal reduction.

If every (1×2)-matrix ((2×1)-matrix) over a ring R has a canonical diagonal reduction then R is called a **right (left) Hermitian ring**. A ring which is both right and left Hermitian is called an **Hermitian ring**. Obviously, a commutative right (left) Hermitian ring is an Hermitian ring. We note that a right Hermitian ring is a ring in which every finitely generated right ideal is principal.

Theorem 3.2

Let R be a right quasi-duo elementary divisor ring. Then for any $a \in R$ there exists an element $b \in R$ such that $RaR = bR = Rb$. If in addition all zero-divisors of R lie in the Jacobson radical, then R is a duo ring. [Zabavsky, Komarnytskii, 1990]

B. V. Zabavsky and M. Ya. Komarnytskii, Distributive elementary divisor domains, *Ukr. Math. J.* **42** (1990) 890–892.

Recall that a **right (left) duo ring** is a ring in which every right (left) ideal is two-sided. A **duo ring** is a ring which is both left and right duo ring.

We have proved the next result.

Theorem 3.3

Let R be an elementary divisor ring which has (right) square stable range 1 and which all zero-divisors of R lie in Jacobson radical of R, then R is a duo ring.

Proposition 3.3

Let R be a Hermitian duo ring. For every $a, b, c \in R$ such that $aR + bR + cR = R$ the following conditions are equivalent:

1) there exist elements $p, q \in R$ such that $paR + (pb + qc)R = R$;
2) there exist elements $\lambda, u, v \in R$ such that $b + \lambda c = vu$, where $uR + aR = R$ and $vR + cR = R$.

Remark 3.1

In Proposition 3.3 we can choose the elements u and v such that $uR + vR = R$.

Proposition 3.4

Let R be an Hermitian duo ring. Then the following conditions are equivalent:

1) R is an elementary divisor duo ring;
2) for every $x, y, z, t \in R$ such that $xR + yR = R$ and $zR + tR = R$ there exists an element $\lambda \in R$ such that $x + \lambda y = vu$, where $vR + zR = R$ and $uR + tR = R$.

Definition

Let R be a duo ring. We say that an element $a \in R \backslash \{0\}$ is **neat** if for any elements $b, c \in R$ such that $bR + cR = R$ there exist elements $r, s \in R$ such that $a = rs$, where $rR + bR = R$, $sR + cR = R$, $rR + sR = R$.

Definition

We say that a duo ring R has **neat range 1** if for every $a, b \in R$ such that $aR + bR = R$ there exists an element $t \in R$ such that $a + bt$ is a neat element.

According to Propositions 3.3, 3.4 and Remark 3.1 we have the following result.

Theorem 3.4

A Hermitian duo ring R is an elementary divisor ring if and only if R has neat range 1.

The term "neat range 1" substantiates the following theorem.

Theorem 3.5

Let R be a Hermitian duo ring. If c is a neat element of R then R/cR is a clean ring.

Recall that a ring R is a **clean ring** if for any $a \in R$ we have $a = u + e$, where u is invertible element and $e^2 = e$.

Taking into account the Theorem 3.3 and Theorem 3.4 we have the following result.

Theorem 3.5

A Hermitian ring R which has (right) square stable range 1 is an elementary divisor ring if and only if R is a duo ring of neat range 1.

Let R be a commutative Bezout ring. The matrix A of order 2 over R is said to be a **Toeplitz matrix** if it is of the form

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix}$$

where $a, b, c \in R$.

Definition

A commutative Hermitian ring R is called a **Toeplitz ring** if for any $a, b \in R$ there exist an invertible Toeplitz matrix T such that $(a, b)T = (d, 0)$ for some element $d \in R$.

Theorem 3.7

A commutative Hermitian ring R is a Toeplitz ring if and only if R is a ring of (right) square range 1.

Theorem 3.8

Let R be a commutative ring of square stable range 1. Then for any row (a, b), where $aR + bR = R$, there exists an invertible Toeplitz matrix

$$T = \begin{pmatrix} a & b \\ x & a \end{pmatrix},$$

where $x \in R$.

Recall that $GE_n(R)$ denotes a group of $n \times n$ elementary matrices over ring R. The following theorem demonstrated that it is sufficient to consider only the case of matrices of order 2 in Theorem 3.7.

Theorem 3.9

Let R be a commutative elementary divisor ring. Then for any $n \times m$ matrix A ($n > 2$, $m > 2$) one can find matrices $P \in GE_n(R)$ and $Q \in GE_m(R)$ such that

$$PAQ = \begin{pmatrix} e_1 & 0 & \dots & 0 & 0 \\ 0 & e_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e_s & 0 \\ 0 & 0 & \dots & 0 & A_0 \end{pmatrix}$$

where e_i is a divisor of e_{i+1}, $1 \leq i \leq s-1$, and A_0 is a $2 \times k$ or $k \times 2$ matrix for some $k \in \mathbb{N}$. [Zabavsky, 2012]

B. V. Zabavsky *Diagonal reduction of matrices over rings* (Mathematical Studies. Monograph Series, volume XVI, VNTL Publishers, Lviv, 2012).

Theorem 3.10

Let R be a commutative elementary divisor ring of square stable range 1. Then for any 2×2 matrix A one can find invertible Toeplitz matrices P and Q such that

$$PAQ = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix},$$

where e_i is a divisor of e_2.

Open Question

Is it true that every commutative Bezout domain of stable range 2 which has (right) square stable range 1 is an elementary divisor ring?

THANK YOU