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RINGS WITH ELEMENTARY REDUCTION OF MATRICES

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We establish necessary and sufficient conditions under which a quasi-Euclidean ring coincides with a ring with elementary reduction of matrices. We prove that a semilocal Bézout ring is a ring with elementary reduction of matrices and show that a 2-stage Euclidean domain is also a ring with elementary reduction of matrices. We formulate and prove a criterion for the existence of solutions of a matrix equation of a special type and write these solutions in an explicit form.

Introduction

The problem of reduction of matrices to the canonical diagonal form by elementary transformations was investigated by Gauss, Smith, van der Waerden, etc. In a more general form, the problem of complete description of both commutative and noncommutative rings over which an arbitrary matrix is reduced to the canonical diagonal form by elementary transformations was formulated in [1].

In the present paper, we establish necessary and sufficient conditions under which a quasi-Euclidean ring coincides with a ring with elementary reduction of matrices. We prove that a semilocal ring is a ring with elementary reduction of matrices and show that a 2-stage Euclidean domain is also a ring with elementary reduction of matrices.

The following fact should also be noted: In the algebraic K-theory, an important role is played by a K_1-functor that associates a certain ring with a Whitehead group. In the commutative case, a Whitehead group decomposes into the direct sum of the group of units of a ring and the quotient group of a special linear group by the subgroup generated by elementary matrices [2, p. 38]. In many interesting cases, the special linear group is generated by elementary matrices (i.e., matrices that differ from the unit matrix by the presence of a single nonzero element outside the main diagonal) and, hence, the Whitehead group is isomorphic to the group of units of a ring. In the present paper, we study precisely these commutative rings on the basis of investigation of rings with elementary reduction of matrices.

Definitions and Assumptions

A ring R is understood as a commutative ring with nonzero unit element and $\mathcal{U}(R)$ is understood as the group of invertible elements of this ring. Denote by (a, b) the greatest common divisor of elements $a, b \in R$. We denote the set of all maximal ideals of the ring R that contain an element a by $\operatorname{mspec}(a)$ and the Jacobson radical by $\mathcal{J}(R)$. Note that the term *semilocality* (a finite number of maximal ideals) does not imply chain conditions. The ring of square matrices of order n with elements of the ring R is denoted by R_n, and the *trace* and *determinant* of a matrix $A \in R_n$ are denoted by $\operatorname{tr} A$ and $\det A$, respectively.

An *elementary matrix* with elements of a ring R is understood as a square matrix of one of the following types [3]:

(i) a diagonal matrix with invertible elements on the main diagonal;

(ii) a matrix that differs from the unit matrix by the presence of a certain nonzero element outside the main diagonal;

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(iii) a permutation matrix, i.e., a matrix obtained from the unit matrix by the permutation of certain rows and columns of the latter.

A group generated by elementary matrices of type (ii) of order n (i.e., by matrices that differ from the unit matrix by the presence of a certain nonzero element outside the main diagonal) is called a *group of elementary matrices* $\mathrm{GE}_n(R)$. We denote by $\mathrm{SL}_n(R)$ a *special linear group*, i.e., a group of matrices of order n whose determinants are equal to 1.

Matrices A and B with elements of a ring R are called *elementarily equivalent* (which is denoted by $A \overset{e}{\sim} B$) if there exist matrices $P_1, \ldots, P_k, Q_1, \ldots, Q_s$ of the correspondent orders that are elementary over R and such that $P_1 \ldots P_k A Q_1 \ldots Q_s = B$. A matrix A *admits elementary reduction* if it is elementarily equivalent to a canonical diagonal matrix $\operatorname{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r, 0, \ldots, 0)$, where $\varepsilon_i R \cap R\varepsilon_i \supseteq R\varepsilon_{i+1}R$, $i = 1, 2, \ldots, r-1$ (generally speaking, a diagonal matrix is understood as a rectangular matrix all elements of which located outside the main diagonal are equal to zero). If an arbitrary matrix over R admits elementary reduction, then R is called a *ring with elementary reduction of matrices* [1]. Rings with elementary reduction of matrices differ from *elementary divisor rings* [4] by the fact that the matrices $P_1, \ldots, P_k, Q_1, \ldots, Q_s$ in their definition are not only invertible but also elementary. It is clear that a ring with elementary reduction of matrices is an elementary divisor ring. However, an elementary divisor ring is not necessarily a ring with elementary reduction of matrices. As an example, one can consider the ring $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$ [1, 5], which, in particular, is a principal ideal ring, but it is not quasi-Euclidean.

A ring R is called *elementarily principal* if, for any $a, b \in R$, there exist $c \in R$ and $M \in \mathrm{GE}_2(R)$ such that $(a, b)M = (c, 0)$ [5].

If we require only the invertibility of the matrix M in this definition, then we arrive at the definition of right Hermite ring, i.e., if, for any elements $a, b \in R$, there exist $c \in R$ and an invertible matrix $M \in R_2$ such that $(a, b)M = (c, 0)$, then the ring R is called a *right Hermite ring. Left Hermite rings* can be defined by analogy. In the commutative case, these classes of rings coincide [4].

Criterion of Hermiticity (Theorem 3 in [6])**.** *A commutative ring R is an Hermite ring if and only if, for any $a, b \in R$, there exist $a_0, b_0, d \in R$ such that $a = a_0 d$, $b = b_0 d$, and $(a_0, b_0) = 1$.*

A ring in which any finitely generated ideal is principal is called a *Bézout ring.* It is clear that an Hermite ring is a Bézout ring [4].

A commutative ring R is called a *ring of stable rank one* if, for any relatively prime $a, b \in R$, there exists an element $t \in R$ such that $a + bt$ is an invertible element of the ring R.

Quasi-Euclidean Rings

A *quasialgorithm* determined on a ring R is understood as a function $\varphi : R \times R \to W$ (W is a certain ordinal) such that, for any $a, b \in R$ $(b \neq 0)$, there exist $q, r \in R$ such that $a = bq + r$ and $\varphi(b, r) < \varphi(a, b)$. A ring R is called *quasi-Euclidean* [5] if there exist a certain ordinal W and a *quasialgorithm* $R \times R \to W$. Examples of quasi-Euclidean rings are Euclidean rings, valuation rings, and regular rings [5].

We need the following well-known statements:

Theorem 1 (Theorem 8 in [5])**.** *The class of quasi-Euclidean rings coincides with the class of elementarily principal rings.*

Theorem 2 (Theorem 17 in [5])**.** *Suppose that R is a ring and, for any $x \in R$, the annihilator $\operatorname{Ann}(x)$ is generated by an idempotent. Then the following assertions are equivalent:*

(i) R *is quasi-Euclidean;*

(ii) R *is a Bézout ring and* $\mathrm{GE}_n(R) = \mathrm{SL}_n(R)$ *for a natural number* $n \geq 2$.

Remark 1. Let $A = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in R_2$, where R is a quasi-Euclidean ring. By virtue of Theorem 1, the ring R is elementarily principal. Hence, for the elements $x, y \in R$, there exist $a \in R$ and an elementary matrix $M \in \mathrm{GE}_2(R)$ such that $(x, y)M = (a, 0)$. Therefore,

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} M = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}.$$

Thus, it suffices to consider a triangular matrix of the form $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ instead of a matrix of the form $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ over the quasi-Euclidean ring.

Proposition 1. *A quasi-Euclidean ring is an Hermite ring.*

To prove this statement, it suffices to note that a quasi-Euclidean ring is elementarily principal (Theorem 1) and an elementarily principal ring is an Hermite ring.

Proposition 2. *A quasi-Euclidean ring* R *is a ring with elementary reduction of matrices if and only if a matrix of the form* $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, *where* $aR + bR + cR = R$, *admits elementary reduction.*

Proof. The necessity is obvious.

To prove the sufficiency, we consider the case where $aR + bR + cR = dR$, where $d \notin \mathcal{U}(R)$. In view of the criterion of Hermiticity, there exist elements $a_1, b_1, c_1 \in R$ such that $a = a_1 d$, $b = b_1 d$, $c = c_1 d$, and $a_1 R + b_1 R + c_1 R = R$. Then $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix}$. Since the matrix $\begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix}$ admits elementary reduction and $\operatorname{diag}(d, d)$ belongs to the center of R_2, the matrix $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ also admits elementary reduction. The proof is completed by induction on the order of matrices.

Theorem 3. *A quasi-Euclidean ring* R *any noninvertible element of which belongs to at most countable set of maximal ideals is a ring with elementary reduction of matrices.*

Proof. According to Proposition 2, to prove Theorem 3 it suffices to consider matrices $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, where $aR + bR + cR = R$. If $a \in \mathcal{U}(R)$, then

$$\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b & c \end{pmatrix} \overset{e}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}.$$

Thus, the matrix A admits elementary reduction. Assume that $a \notin \mathcal{U}(R)$, i.e., the set of maximal ideals that contain a is not empty $(\operatorname{mspec}(a) \neq \varnothing)$. We set $\operatorname{mspec}(a) = \{\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n, \ldots\}$. Then we can assume that

$b \notin \mathcal{M}_1$ (if $b \in \mathcal{M}_1$, then $(b+c) \notin \mathcal{M}_1$ because $aR + bR + cR = R$ and the element b can be replaced by the element $(b+c)$ by elementary transformations of columns).

Since the ring R is elementarily principal (Theorem 1), there exists an elementary matrix P_1 such that $P_1 A = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$, where $aR + bR = a_1 R$.

If $a_1 \in \mathcal{U}(R)$, then $P_1 A$ admits elementary reduction. Therefore, we can assume that $a_1 \in \mathcal{M}_2$ to within notation. Then $b_1 \notin \mathcal{M}_2$ (or $(b_1 + c_1) \notin \mathcal{M}_2$ because $a_1 R + b_1 R + c_1 R = R$). Since the ring R is elementarily principal, there exists an elementary matrix Q_1 such that $P_1 A Q_1 = \begin{pmatrix} a_2 & 0 \\ b_2 & c_2 \end{pmatrix}$, where $a_1 R + b_1 R = a_2 R$.

Continuing this process, we obtain a collection of matrices of the form

$$P_k A Q_k = \begin{pmatrix} a_i & * \\ * & * \end{pmatrix},$$

associated with the following chain of ideals:

$$aR \subset a_1 R \subset a_2 R \subset \ldots \subset a_i R \subset \ldots; \tag{1}$$

furthermore, $a_i \notin \mathcal{M}_i$.

Denote $I = \bigcup_i a_i R$. Assume that $I \neq R$. Then there exists a maximal ideal $\mathcal{M}$ such that $I \subset \mathcal{M}$. Since $aR \subset I$, we have $\mathcal{M} = \mathcal{M}_s$, where $\mathcal{M}_s \in \operatorname{mspec}(a)$. This is impossible because there exists an ideal $a_s R$ from chain (1) such that $a_s \notin \mathcal{M}_s$. Thus, $I = R$, i.e., chain (1) is finite and, hence, there exist matrices P_n, Q_n (which are finite products of elementary matrices) such that $P_n A Q_n$ is a canonical diagonal matrix.

Corollary 1. *A semilocal quasi-Euclidean ring is a ring with elementary reduction of matrices.*

Corollary 2. *A quasi-Euclidean ring the set of maximal ideals of which is at most countable is a ring with elementary reduction of matrices.*

Lemma 1. *Any invertible matrix over a quasi-Euclidean ring R is a finite product of elementary matrices.*

Proof. First, we prove that any invertible matrix over a quasi-Euclidean ring can be reduced to the diagonal form by elementary transformations. We prove this for a matrix of the second order.

Let $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ be a matrix invertible over R. Then $(a, b) = 1$ and, since a quasi-Euclidean ring is elementarily principal (Theorem 1), there exists a matrix $Q \in \mathrm{GE}_2(R)$ such that $(a, b)Q = (1, 0)$, i.e.,

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} Q = \begin{pmatrix} 1 & 0 \\ b_1 & c_1 \end{pmatrix}.$$

Assume that $P = \begin{pmatrix} 1 & 0 \\ -b_1 & 1 \end{pmatrix}$, $P \in \mathrm{GE}_2(R)$. We have

$$PAQ = \begin{pmatrix} 1 & 0 \\ -b_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_1 & c_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & c_1 \end{pmatrix},$$

which was to be proved.

Thus, if A is an arbitrary invertible matrix over the quasi-Euclidean ring R, then there exist matrices P and Q (which are finite products of elementary matrices) such that

$$PAQ = F,$$

where F is a diagonal matrix. It is clear that the matrix F is invertible. Since the determinant of the matrix F is equal to the product of diagonal elements, all these diagonal elements are invertible. Thus, F is an elementary matrix [of type (i)]. Since

$$A = P^{-1}FQ^{-1},$$

we conclude that any invertible matrix over a quasi-Euclidean ring is a finite product of elementary matrices.

Theorem 4. *A semilocal Bézout ring is a ring with elementary reduction of matrices.*

Proof. Assume that R is a semilocal Bézout ring and elements $a, b \in R$ satisfy the equality $aR + bR = R$. Then

$$\operatorname{mspec}(a) \cap \operatorname{mspec}(b) = \varnothing. \tag{2}$$

We denote by r an element of the ring R that belongs to all maximal ideals of the ring R except maximal ideals of the set $\operatorname{mspec}(a)$ (since the ring R is semilocal, such an element r exists). It is clear that

$$\operatorname{mspec}(r) \cap \operatorname{mspec}(a) = \varnothing. \tag{3}$$

We consider the element $a + br \in R$. Assume that $a + br \in \mathcal{M}$, where $\mathcal{M}$ is a maximal ideal of the ring R. The following cases are possible:

1. $a \in \mathcal{M}$ and $b \in \mathcal{M}$, which contradicts condition (2).

2. $a \in \mathcal{M}$ and $r \in \mathcal{M}$, which contradicts condition (3).

Thus, our assumption is not true. Hence, $a + br = u \in \mathcal{U}(R)$. Then

$$(a, b)\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} = (u, b) \overset{e}{\sim} (u, 0).$$

Thus, the ring R is elementarily principal and, hence, it is quasi-Euclidean. Then, by virtue of Corollary 1, a semilocal Bézout ring is a ring with elementary reduction of matrices.

We now impose the following condition on the ring R:

Condition 1. *We assume that, for* $a, b \in R (a \notin \mathcal{J}(R))$, *there exists* $m \in R$ *such that* $(b, m) = 1$ *and, for any* $n \in R$ *such that* $(n, a) \neq 1$ *and* $(n, b) = 1$, *we have* $(n, m) \neq 1$.

In [7], the following statement was proved:

Theorem 5 (Corollary 2.6 in [7]). *An Hermite ring R whose elements satisfy Condition 1 is an elementary divisor ring.*

We use Theorem 5 in the proof of the following statement, in which we slightly modify Condition 1:

Theorem 6. *Suppose that* R *is an Hermite ring and, for any* $a, b \in R$, $b \neq 0$, *there exists* $s \in R$ *such that* $\operatorname{mspec}(s) = \operatorname{mspec}(a) \setminus \operatorname{mspec}(b)$. *Then* R *is a ring with elementary reduction of matrices.*

Proof. Let elements $a, b \in R$ be such that $aR + bR = R$. It is clear that $\operatorname{mspec}(a) \cap \operatorname{mspec}(b) = \varnothing$. Under the conditions of Theorem 6, there exists an element $r \in R$ that belongs to all maximal ideals of the ring R except the maximal ideals of the set $\operatorname{mspec}(a)$. Consequently, $\operatorname{mspec}(r) = \operatorname{mspec}(0) \setminus \operatorname{mspec}(a)$. It is obvious that $\operatorname{mspec}(r) \cap \operatorname{mspec}(a) = \varnothing$.

We consider the element $a + br \in R$. By analogy with the proof of Theorem 4, one can easily prove that $a + br \in \mathcal{U}(R)$, i.e., the ring R is quasi-Euclidean. Then, by virtue of Theorem 5 and Lemma 1, R is a ring with elementary reduction of matrices.

2-Stage Euclidean Domains

Let $a \in R \setminus 0$ and $b \in R$. An *n-stage division chain* [8] is understood as a sequence of equalities $b = q_1 a + r_1$, $a = q_2 r_1 + r_2, \ldots,$ $r_{n-2} = q_n r_{n-1} + r_n$.

Let R be a commutative ring without zero divisors. We assume that there exists a function $\mathcal{N}\colon R \to \mathbb{Z}$ such that $\mathcal{N}(0) = 0$, $\mathcal{N}(a) > 0$ for $a \neq 0$, and $\mathcal{N}(ab) \geq \mathcal{N}(a)$ for any $a, b \in R \setminus 0$. The function $\mathcal{N}$ thus defined is called a *norm* on R. It should be noted that the condition that $\mathcal{N}(ab) \geq \mathcal{N}(a)$ for $a, b \in R \setminus 0$ is superfluous. Indeed, if $\mathcal{N}\colon R \to \mathbb{Z}$ is a function such that $\mathcal{N}(0) = 0$ and $\mathcal{N}(a) > 0$ for any $a \neq 0$, then, by choosing $\mathcal{N}_1(a) = \min\{\mathcal{N}(ab) \mid b \in R \setminus 0\}$, one can easily establish that the function $\mathcal{N}_1$ is a norm on R.

A commutative ring R without zero divisors with a given norm $\mathcal{N}$ is called an *n-stage Euclidean domain* [8] with respect to the norm $\mathcal{N}$ if, for any elements $a, b \in R$, $a \neq 0$, there exists a k-stage division chain for some $k \leq n$ such that $\mathcal{N}(r_k) < \mathcal{N}(a)$. A commutative ring R without zero divisors is called *w-stage Euclidean* [8] if, for any elements $a, b \in R$, $a \neq 0$, there exists a k-stage division chain for some k such that $\mathcal{N}(r_k) < \mathcal{N}(a)$. It is obvious that a 1-stage Euclidean domain is a Euclidean domain.

A commutative ring R without zero divisors is called a GE_n*-domain* if any invertible matrix over R is generated by elementary matrices of type (ii) of order n [3].

Below, we present several well-known results and their corollaries.

Proposition 3 (Proposition 23 in [5]). *An integral domain* R *is quasi-Euclidean if and only if* R *is a* w*-stage Euclidean domain.*

Proposition 4 (Proposition 14 in [8]). *A commutative ring without zero divisors is a* w*-stage Euclidean domain if and only if it is a Bézout* GE_2*-domain.*

Assume that R is a domain with elementary reduction of matrices. Then, according to the definition of R, any row admits elementary reduction. By virtue of Theorem 2, R is a quasi-Euclidean domain and, in view of Proposition 3, R is an n-stage Euclidean domain, i.e., the following statement is true:

Proposition 5. *An arbitrary domain with elementary reduction of matrices is an* n*-stage Euclidean domain.*

In what follows, we consider 2-stage Euclidean domains. For this reason, we now define these domains more exactly. A commutative ring R without zero divisors is called a *2-stage Euclidean domain with respect to the norm* $\mathcal{N}$ if, for any $b \in R$ and $a \in R \setminus 0$, one of the following conditions is satisfied:

(a) there exist $q, r \in R$ such that $b = aq + r$, where $\mathcal{N}(r) < \mathcal{N}(a)$;

(b) there exist $q_1, r_1, q_2, r_2 \in R$ such that $b = aq_1 + r_1$ and $a = r_1 q_2 + r_2$, where $\mathcal{N}(r_2) < \mathcal{N}(a)$.

Proposition 6. *A Bézout domain of stable rank* 1 *is a* 2*-stage Euclidean domain.*

Proof. It is obvious that it suffices to prove Proposition 6 in the case of two relatively prime elements. Let $a, b \in R$ and $aR + bR = R$. By virtue of the definition of R, there exist elements $t \in R$ and $u \in \mathcal{U}(R)$ such that $a - bt = u$. This yields $a = bt + u$ and $b = uu^{-1}b + 0$. According to [8], R is a 2-stage Euclidean domain.

Theorem 7. *A* 2*-stage Euclidean domain is a domain with elementary reduction of matrices.*

Proof. According to Proposition 4, it suffices to prove Theorem 7 for matrices of the second order with relatively prime elements. Assume that A is a matrix of this type in the class of elementarily equivalent matrices and its element located at the intersection of the first row and the first column is a nonzero element of the least norm. We assume that a is precisely this element to within notation, i.e., $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let $b \neq 0$. According to the definition of R, we have $b = aq_1 + r_1$ and $a = r_1 q_2 + r_2$, where $\mathcal{N}(r_2) < \mathcal{N}(a)$ ($\mathcal{N}$ is the norm of the domain R). The case $r_2 \neq 0$ is not interesting because, in this case, for the matrix A, there exists an elementarily equivalent matrix the element of which located at the intersection of the first row and the first column is a nonzero element whose norm is smaller than the norm of the element a, which contradicts the choice of the element a. Thus, let $b = aq + r$ and $a = rs$ for certain $r, s \in R$. Then we obtain the chain of elementarily equivalent matrices

$$A \overset{e}{\sim} \begin{pmatrix} a & b - aq \\ c & d - cq \end{pmatrix} = \begin{pmatrix} a & r \\ * & * \end{pmatrix} \overset{e}{\sim} \begin{pmatrix} r & 0 \\ * & * \end{pmatrix} = B.$$

Since $rR \subset aR$, according to the definition of the norm $\mathcal{N}$ and the choice of the element a we have $\mathcal{N}(r) = \mathcal{N}(a)$. Thus, we have reduced the matrix A to the triangular form $\begin{pmatrix} r & 0 \\ * & * \end{pmatrix}$ by elementary transformations. Thus, the case $b \neq 0$ is not important. Hence, to within notation, we can assume that $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, where a is a nonzero element of R with the least norm among all elements of matrices elementarily equivalent to A.

It should be noted that $aR + bR + cR = R$. By virtue of the restrictions imposed on R, we have $b = aq_1 + r_1$ and $a = r_1 q_2 + r_2$, where $\mathcal{N}(r_2) < \mathcal{N}(a)$. As in the case of columns, the case $r_2 \neq 0$ is obvious. Hence, let $r_2 = 0$. Then

$$A \overset{e}{\sim} \begin{pmatrix} a & 0 \\ r_1 & c \end{pmatrix} = \begin{pmatrix} r_1 q_2 & 0 \\ r_1 & c \end{pmatrix} = B$$

and $r_1 q_2 R + r_1 R + cR = r_1 R + cR = R$. Since a 2-stage Euclidean domain is a Bézout GE_2-domain, for the row (r_1, c) there exist elementary matrices $P_1, \ldots, P_n$ such that $(r_1, c)P_1 \ldots P_n = (1, 0)$. Therefore,

$$BP_1 \ldots P_n = \begin{pmatrix} \alpha & \beta \\ 1 & 0 \end{pmatrix} = C.$$

Then the matrix C and, hence, the matrix A obviously admit elementary reduction.

Remark 2. Examples of 2-stage Euclidean domains were considered in [5, 8]. Of interest is the example of the ring of all integral algebraic numbers. This ring is a Bézout domain [9]. Furthermore, it is not a domain of principal ideals because it does not contain atoms, i.e., it is not a Euclidean domain with respect to a certain norm. This ring is a domain of stable rank 1 [8]. By virtue of Proposition 4 and Theorem 7, the ring of all integral algebraic numbers is a ring with elementary reduction of matrices.

Some Applications of the Results Obtained

Theorem 8. *For a quasi-Euclidean domain* R, *the following assertions are equivalent:*

(i) R *is a ring with elementary reduction of matrices;*

(ii) for an arbitrary matrix $A \in R_2$ *the greatest common divisor of all elements of which is equal to unit, there exists a proper (i.e., nonzero and nonunit) idempotent in the right ideal* AR_2;

(iii) the matrix equation $XAX = X$, *where* $X, A \in R_2$, *has a nonzero solution and the greatest common divisor of all elements of the matrix* A *is equal to unit.*

Proof. We prove Theorem 8 according to the following scheme: (i) ⇒ (ii) ⇒ (iii) ⇒ (ii) ⇒ (i).

(i) ⇒ (ii). Let $aR + bR + cR = R$, where $a, b, c \in R$. Consider the matrix

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R_2.$$

Since R is a ring with elementary reduction of matrices, for the matrix A there exist invertible matrices $P, Q \in R_2$ (which are finite products of elementary matrices) such that $PAQ = \begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix}$. Then

$$\left(AQ\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}P\right)^2 = AQ\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}PAQ\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}P = AQ\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}P = AQ\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}P.$$

We see that $AQ\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}P$ is the required idempotent in the ideal AR_2.

(ii) ⇒ (iii). Let an idempotent $E = E^2 \in AR_2$. Then $E = AB$, where $A \in R_2$. Consider the product

$$BABABAB = (BAB)(ABAB) = (BAB)AB = B(ABAB) = BAB.$$

Hence, by setting $X = BAB$, we get $XAX = X$. Thus, the equation $XAX = X$ has a nonzero solution.

(iii) ⇒ (ii). Let the equation $XAX = X$ have a nonzero solution. Then, multiplying this equation from the left by A, we obtain $AXAX = AX$. This implies that any right ideal AR_2 has a proper idempotent.

(ii) $\Rightarrow$ (i). Consider $a, b, c \in R$ such that $aR + bR + cR = R$. Assume that $\begin{pmatrix} r & k \\ s & t \end{pmatrix} \in R_2$ is such that

$$F = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} r & k \\ s & t \end{pmatrix} = \begin{pmatrix} ar + bs & ak + bt \\ cs & ct \end{pmatrix}$$

is a proper idempotent.

It was proved in Lemma 1 in [10] that the matrix $A \in R_2$, where R is an integral domain, is a proper idempotent if and only if $\det M = 0$ and $\operatorname{tr} M = 1$. Taking into account this fact and the equalities $\operatorname{tr} F = 1$ and $\det F = 0$, we obtain $ar + bs + ct = 1$ and $sk = rt$ ($\mathrm{rt}R \subseteq sR$), respectively.

Let $dR = rR + sR$. Assume that $r = dp$ and $s = dq$, where $p, q \in R$ and $(p, q) = 1$. Then, by using the equality $sk = rt$, we get $dpt = dqk$, i.e., $pt = qk$. This implies that q is a divisor of t and, hence, there exists $m \in R$ such that $t = qm$.

We set

$$P = \begin{pmatrix} d & m \\ -cq & ap + bq \end{pmatrix}, \quad Q = \begin{pmatrix} p & -bd - cm \\ q & ad \end{pmatrix}.$$

It is easy to verify that $\det P = \det Q = 1$. By virtue of Theorem 2, the special linear group $\mathrm{SL}_2(R)$ coincides with the group of elementary matrices $\mathrm{GE}_2(R)$. Hence, P and Q are invertible matrices that are finite products of matrices elementary over R.

Consider the product

$$PAQ = \begin{pmatrix} d & m \\ -cq & ap + bq \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} p & -bd - cm \\ q & ad \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & ac \end{pmatrix}.$$

We see that the matrix A and, hence, an arbitrary matrix over R admit elementary reduction. Therefore, $RXAX = X$ is a ring with elementary reduction of matrices.

It remains to write a solution of the matrix equation $XAX = X$ in an explicit form. It is known that $X = BAB$, where AB is a proper idempotent. Hence,

$$X = \begin{pmatrix} r & k \\ s & t \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} r & k \\ s & t \end{pmatrix} = \begin{pmatrix} r & k \\ s & t \end{pmatrix},$$

where $ar + bs + ct = 1$ and $sk = rt(\mathrm{rt}\, R \subseteq sR)$. Theorem 8 is proved.

Consider the group $\mathrm{SL}_2(R)$, where R is a ring with elementary reduction of matrices. In view of Theorem 2, R is a Bézout domain in which $\mathrm{GE}_2(R) = \mathrm{SL}_2(R)$, i.e., any matrix of the second order over the ring R the determinant of which is equal to unit can be represented in the form of a finite product of matrices $\begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}$ and $\begin{pmatrix} b & 1 \\ -1 & 0 \end{pmatrix}$.

To prove this statement, we note that the group $\mathrm{GE}_2(R)$ is generated by matrices of the form $F(a) = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}$, where $a \in R$ [5]. Indeed,

$$F(a) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in \mathrm{GE}_2(R),$$

$$F^{-1}(a) = F(0)F(-a)F(0) \in \mathrm{GE}_2(R),$$

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = (F(0))^3 F(a), \qquad \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = F(-a)(F(0))^3.$$

As a consequence, we obtain the following statements:

Theorem 9. *Let R be a 2-stage Euclidean domain. Then the group $\mathrm{SL}_2(R)$ is generated by matrices of the form $\begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}$ and $\begin{pmatrix} b & 1 \\ -1 & 0 \end{pmatrix}$, where $a, b \in R$.*

Theorem 10. *The group $\mathrm{SL}_2(R)$, where R is a Bézout domain of stable rank 1, is generated by matrices of the form $\begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}$ and $\begin{pmatrix} b & 1 \\ -1 & 0 \end{pmatrix}$, where $a, b \in R$.*

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